Integrable hydrodynamic chains

Maxim V.Pavlov

February 8, 2008

Abstract

A new approach for derivation of Benney-like momentum chains and integrable hydrodynamic type systems is presented. New integrable hydrodynamic chains are constructed, all their reductions are described and integrated. New (2+1) integrable hydrodynamic type systems are found.

MSC: 37K18, 37K25, 37K35

keywords: Euler-Darboux-Poisson equation, hydrodynamic type system, momentum chain, conservation law, commuting flow, integrable reduction.

1 Introduction

The integrable hydrodynamic chain

$$\partial_t A_k = \partial_x A_{k+1} + kA_{k-1}A_{0,x}, \qquad k = 0,\, 1,\, ... \tag{1}$$

for the first time was introduced by D.J. Benney in a theory of finite-depth fluid (see [5]). Here moments A_k are infinite number of field variables. Later, it was shown that these moments satisfy a dispersionless limit of KP hierarchy determined by the Sato pseudo-differential operator

$$\hat{L} = \partial_x + A_0\partial_x^{-1} + A_1\partial_x^{-2} + ...,$$

which in dispersionless limit is reduced to

$$\lambda = \mu + \frac{A_0}{\mu} + \frac{A_1}{\mu^2} + ... \tag{2}$$

The Benney momentum chain can be written in equivalent form (see [18])

$$\lambda_t - \mu\lambda_x = \frac{\partial\lambda}{\partial\mu}[\mu_t - \partial_x(\frac{\mu^2}{2} + A_0)]. \tag{3}$$

If $\lambda =$ const, then μ is a generating function (with respect to the parameter λ) of the conservation law densities

$$\mu_t = \partial_x(\frac{\mu^2}{2} + A_0). \tag{4}$$

In the case, when several first moments A_k are functionally-independent ($k = 0, 1, ..., N - 1$), the corresponding hydrodynamic reductions ("hydrodynamic" reduction means that all higher moments A_k ($k = 1, 2, ...$) could not depend of any derivatives of lower moments A_k ($k = 0, 1, ..., N - 1$); in opposite case, such reductions one can call as "differential" reductions. In this article we concentrate attention at hydrodynamic reductions only in spirit of [19]) are the hydrodynamic type systems written in Riemann invariants r^i

$$r_t^i = \mu_i(\mathbf{r})r_x^i, \qquad i = 1, 2, ..., N, \tag{5}$$

i.e. hydrodynamic type system has diagonal form in these field variables and there is no summation over each repeated index, see for instance [36]. The Riemann invariants r^i and the characteristic velocities $\mu_i(\mathbf{r})$ are determined by conditions

$$r^i = \mu_i + \frac{A_0}{\mu_i} + \frac{A_1}{\mu_i^2} + \frac{A_2}{\mu_i^3} + ..., \qquad\qquad 1 = \frac{A_0}{\mu_i^2} + 2\frac{A_1}{\mu_i^3} + 3\frac{A_2}{\mu_i^4} + ...$$

(see (2) and (3)). These hydrodynamic type systems are integrable too (all moments A_k are some functions of r^i, which are determined by compatibility conditions with whole Benney momentum chain, see [19]). These hydrodynamic type systems have the same generating functions of conservation laws (see (4)) and the commuting flows (see [31]; "commuting flows" means, that the Riemann invariants r^i simultaneously are functions of infinite number of independent variables t_k, $k = 0, 1, ...$, here $t_0 \equiv x$, $t_1 \equiv t$.)

$$\mu(\lambda)_{\tau(\tilde{\lambda})} = \partial_x \ln[\mu(\lambda) - \mu(\tilde{\lambda})], \tag{6}$$

where

$$\partial_{\tau(\tilde{\lambda})} = \partial_{t_0} + \frac{1}{\tilde{\lambda}}\partial_{t_1} + \frac{1}{\tilde{\lambda}^2}\partial_{t_2} + ...$$

Moreover, the generating function (with respect to the parameter $\tilde{\lambda}$) of solutions for any reduction (5) can be found by the Tsarev generalized hodograph method ([36], also see [31])

$$x + \mu_i(\mathbf{r})t = \frac{1}{\mu_i(\mathbf{r}) - \mu(\tilde{\lambda})}, \qquad i = 1, 2, ..., N \tag{7}$$

Thus, if some hydrodynamic type system is recognized as a reduction of the Benney momentum chain, it means that this system has most properties of the Benney momentum chain.

The idea presented in this paper is the following: if one can introduce the moments A_k for given integrable hydrodynamic type system (5), then **one can ignore the origin** (i.e. given hydrodynamic type system) of this *hydrodynamic chain*

$$\partial_t A = F(\mathbf{A})A_x,$$

where A is an infinite-number component vector, $F(\mathbf{A})$ is an infinite-number component matrix. The next step is a description of all possible integrable hydrodynamic reductions (one of them, of course, must be the original hydrodynamic type system (5))

$$r_t^i = V_i(\mathbf{r})r_x^i, \qquad i = 1, 2, ..., M, \tag{8}$$

where M is not connected with N (see (5)), and V_i satisfy some nonlinear system of PDE's (see below). Also, we assume that $V_i \neq V_k$ for any $i \neq k$ (this is a necessary condition for the application of the Tsarev generalized hodograph method). Thus, every hydrodynamic chain constructed in this way can be regarded as a *huge box* for some variety of the integrable hydrodynamic type systems.

For instance, the particular case of gas dynamics

$$u_t = \partial_x[\frac{u^2}{2} + \frac{\eta^{\gamma-1}}{\gamma-1}], \qquad \eta_t = \partial_x(u\eta), \tag{9}$$

for $\gamma = 2$ (shallow water equations):

$$u_t = \partial_x[\frac{u^2}{2} + \eta], \qquad \eta_t = \partial_x(u\eta) \tag{10}$$

satisfies for Benney momentum chain (1) if one introduces the moments $A_k = u^k\eta$.

It is easy to check, that the Benney momentum chain has a more general (the Zakharov) reduction $A_k = \sum\limits_{i=1}^{N} u_i^k \eta_i$ (see [38]), which create a dispersionless limit of vector nonlinear Shrodinger equation (VNLS)

$$\partial_t u_i = \partial_x[\frac{u_i^2}{2} + \sum_{k=1}^{N}\eta_k], \qquad \partial_t\eta_i = \partial_x(u_i\eta_i), \qquad i = 1,\, 2,\, ...,\, N \tag{11}$$

Remark 1 *Infinite series (2) under this Zakharov reduction yields a more compact expression (see [38])*

$$\lambda = \mu + \sum_{k=1}^{N}\frac{\eta_k}{\mu - u_k}. \tag{12}$$

It is easy to check that the dispersionless limit of VNLS satisfies equation (3) with respect to equation of Riemann surface (12).

Obviously, in both above-mentioned cases, corresponding hydrodynamic type systems (10) and (11) have the same generating functions of conservation law densities (4) and commuting flows (6) as whole Benney momentum chain (1). Here we demonstrate this approach on an example of a new hydrodynamic chain, which contains some important reductions well known in mathematics, fluid dynamics, nonlinear optics, biology and chemistry.

The main classification problem in the theory of integrable hydrodynamic type systems can be re-formulated as the problem of description of all possible integrable hydrodynamic chains. For the simplicity, any hierarchy of the hydrodynamic chains can be written in a conservative form (see, for instance, (30) below)

$$\partial_{t_n} A_k = \partial_x F_k(A_{k+n},\, A_{k+n-1},\, \dots\, ,\, A_0), \qquad k,\, n = 0,\, 1,\, ...$$

If one can classify all possible functions F_k, it would mean that all the hydrodynamic type systems embedded in such hydrodynamic chains by different reductions are classified too. In simplest case ($N=2$)

$$\partial_t A_0 = \partial_x F_0(A_0,\, A_1), \qquad \partial_t A_1 = \partial_x F_1(A_0,\, A_1,\, A_2),$$

where $A_2(A_0,\, A_1)$ is conservation law density of reduced hydrodynamic type system. Then governing equation for function A_2 is quasilinear

$$f_\upsilon w_{uu} = [f_u - \varphi_\upsilon - \varphi_w w_\upsilon] w_{u\upsilon} + [\varphi_u + \varphi_w w_u] w_{\upsilon\upsilon},$$

where

$$u \equiv A_0, \quad \upsilon \equiv A_1, \quad w \equiv A_2, \quad f \equiv F_0, \quad \varphi \equiv F_1.$$

When $F_0 = \upsilon$, $F_1 = w(u,\, \upsilon) - u^2/2$, this is $2-$component reduction of the Benney momentum chain (see [19])

$$w_{uu} = -w_\upsilon w_{u\upsilon} + (w_u - u) w_{\upsilon\upsilon};$$

if $F_0 = \upsilon - u^2$, $F_1 = w(u,\, \upsilon) - u\upsilon$, then corresponding equation

$$w_{uu} = -(u + w_\upsilon) w_{u\upsilon} + (w_u - \upsilon) w_{\upsilon\upsilon},$$

can be solved in parametric form

$$w = \tfrac{1}{6}[A''(s) + B''(r)]^3 + [A''(s) + B''(r)][A'(s) - sA''(s) + B'(r) - rB''(r)] + \\ s^2 A''(s) - 2sA'(s) + 2A(s) + r^2 B''(r) - 2rB'(r) + 2B(r),$$

$$\upsilon = \tfrac{1}{2}[A''(s) + B''(r)]^2 + A'(s) - sA''(s) + B'(r) - rB''(r), \qquad u = A''(s) + B''(r),$$

where $A(s)$ and $B(r)$ are arbitrary functions. Thus, $2-$component reduced (hydrodynamic type) system in Riemann invariants is

$$r_t = (A''(s) + B''(r) + r) r_x, \quad s_t = (A''(s) + B''(r) + s) s_x.$$

This system is natural $2-$parametric generalization of gas dynamics (9) (see below).

The goal of this paper is a complete description of $N-$component generalization of above-mentioned formulas.

In section 2 of this paper, so-called "$\varepsilon-$systems" are introduced. All their properties like conservation laws and commuting flows are described. The corresponding hydrodynamic chain is found by the natural introduction of moments.

In section 3, some properties such transformations between different representations of this hydrodynamic chain are obtained.

In section 4, all possible hydrodynamic reductions are found. Particular and important reductions of this hydrodynamic chain are emphasized.

In section 5, a generating functions of conservation law densities, commuting flows and solutions (by the Tsarev generalized hodograph method) for these hydrodynamic reductions are constructed.

In section 6, a general solution of these hydrodynamic type systems is presented.

In section 7, new $(2+1)$ integrable hydrodynamic type systems are found.

In section 8, another hydrodynamic chain is presented, all its hydrodynamic reductions are described.

In section 9, we discuss some still open problems: Hamiltonian structures and integrable dispersive extensions of hydrodynamic chains and their reductions.

In section 10 (Conclusion), we describe a general situation in theory of hydrodynamic chains.

2 "$\varepsilon-$systems".

This class of integrable hydrodynamic type systems

$$r_t^i = [r^i - \varepsilon \sum_{m=1}^{N} r^m] r_x^i, \qquad i = 1, 2, ..., N, \tag{13}$$

where ε is an arbitrary constant, was established in [32] (also see [16], [29], [30], [33]). These hydrodynamic type systems (13) and its commuting flows (see below) we shall call "$\varepsilon-$systems". The particular case $N = 2$ plays important role in gas dynamics (see (9), where the adiabatic index $\gamma = \frac{3-2\varepsilon}{1-2\varepsilon}$), in field theory ($\gamma = 1$, Born-Infeld equation), in nonlinear optics ($\gamma = 2$, the dispersionless limit of nonlinear Shrodinger equation, see (9)) and in fluid dynamics ($\gamma = 4$, the dispersionless limit of the second commuting flow to the Boussinesq equation). Also, "$\varepsilon-$systems" (for arbitrary N) are well known in differential geometry ($\varepsilon = -1/2$, elliptic coordinates, see, for instance, [30]; dispersionless limit of Coupled KdV, see, for instance, [15]), in soliton theory ($\varepsilon = 1$, some particular solutions of linearly-degenerated systems are multi-gap solutions of KdV, see [13]), in biology and chemistry ($\varepsilon = -1$, chromatography, electrophoresis, isotahophoresis). Moreover, a general solution can be found explicitly (see [29]), for instance, in one-atomic ($\gamma = \frac{5}{3}$, $\varepsilon = -1$), two-atomic ($\gamma = \frac{7}{5}$, $\varepsilon = -2$) gases (see (13)) and their generalization for arbitrary N and arbitrary **integer** ε. Thus, obvious aim is to extend a class of integrable hydrodynamic type systems starting from (13) with preservation of some properties.

The hydrodynamic type system (13) has a generation function μ of conservation law densities; when $\lambda \to \infty$

$$\mu \equiv \prod_{m=1}^{N} (1 - r_m/\lambda)^{-\varepsilon} = 1 + a_1/\lambda + a_2/\lambda^2 + ..., \tag{14}$$

when $\lambda \to 0$ (up to constant multiplier)

$$\mu \equiv \prod_{m=1}^{N} (r_m - \lambda)^{-\varepsilon} = b_0 + b_1\lambda + b_2\lambda^2 + ... \tag{15}$$

The first series (14) is a series of polynomial conservation law densities a_k with respect to Riemann invariants (this is analogue of Kruskal series of conservation law densities for integrable dispersive systems like Korteweg de Vries equation). We shall call them as "higher" (or "positive") conservation law densities correspondingly their homogeneity.

Coefficients b_k we shall call as "lower" (or "negative") conservation law densities (they play role as a "new" conservation law densities appearing under Miura type transformation in theory of integrable dispersive systems).

It is easy to check, that any commuting flow (so, every Riemann invariant r^i is a function of three independent variables x, t, τ)

$$r^i_\tau = w^i_{(\varepsilon)}(\mathbf{r})r^i_x \tag{16}$$

to hydrodynamic type system (13) has velocities

$$w^i_{(\varepsilon)}(\mathbf{r}) = \partial_i h_{(-\varepsilon)}, \tag{17}$$

where $h_{(-\varepsilon)}$ is some conservation law density of "$(-\varepsilon)-$system"

$$r^i_t = [r^i + \varepsilon\sum_{m=1}^N r^m]r^i_x, \qquad i = 1,\, 2,\, ...,\, N.$$

Since, "$\varepsilon-$systems" and "$(-\varepsilon)$-systems" have generating functions of conservation law densities such that

$$\mu_{(\varepsilon)} \cdot \mu_{(-\varepsilon)} = 1, \tag{18}$$

then a generating function of commuting flows to (13) in Riemann invariants is (see (16) and (17))

$$r^i_{\tau(\tilde{\lambda})} = \frac{1}{(1 - r^i/\tilde{\lambda})\tilde{\mu}}r^i_x, \tag{19}$$

and in conservative form (sf. 6) is

$$\mu_{\tau(\tilde{\lambda})} = \frac{\tilde{\lambda}}{\tilde{\lambda} - \lambda}\partial_x(\frac{\mu}{\tilde{\mu}}), \tag{20}$$

where $\tilde{\mu} \equiv \mu(\tilde{\lambda})$ (see (14) and (15)).

Higher commuting flows can be obtained from (see (14))

$$\tilde{\mu} \equiv \prod_{m=1}^N (1 - r_m/\tilde{\lambda})^{-\varepsilon} = 1 + a_1/\tilde{\lambda} + a_2/\tilde{\lambda}^2 + ...$$

and formal series

$$\partial_{\tau(\tilde{\lambda})} = \partial_{t_0} + \frac{1}{\tilde{\lambda}}\partial_{t_1} + \frac{1}{\tilde{\lambda}^2}\partial_{t_1} + ...$$

when $\tilde{\lambda} \to \infty$. The corresponding generating functions of conservation laws are

$$\partial_{t_k}\mu = \partial_x[\mu\sum_{m=0}^k \tilde{a}_m\lambda^{k-m}], \qquad k = 0, 1, 2, ... \tag{21}$$

where

$$\tilde{a}_0 = a_0 = 1, \qquad \tilde{a}_1 = -a_1, \qquad \tilde{a}_n = -a_n - \sum_{m=1}^{n-1}\tilde{a}_m a_{n-k}, \qquad n = 2,\, 3,\, ...$$

The corresponding *higher* commuting flows (in Riemann invariants) are

$$\partial_{t_k} r^i = [\sum_{m=0}^{k} \tilde{a}_m (r^i)^{k-m}] r_x^i, \qquad k = 0, 1, 2, ... \tag{22}$$

The *lower* commuting flows can be obtained from (see (15))

$$\tilde{\mu} \equiv \prod_{m=1}^{N} (r_m - \tilde{\lambda})^{-\varepsilon} = b_0 + b_1 \tilde{\lambda} + b_2 \tilde{\lambda}^2 + ...$$

and formal series

$$\partial_{\tau(\tilde{\lambda})} = \tilde{\lambda} \partial_{t_{-1}} + \tilde{\lambda}^2 \partial_{t_{-2}} + \tilde{\lambda}^3 \partial_{t_{-3}} + ...,$$

when $\tilde{\lambda} \to 0$. The corresponding generating functions of conservation laws are

$$\partial_{t_{-k-1}} \mu = \partial_x [\mu \sum_{m=0}^{k} \tilde{b}_m \lambda^{m-k-1}], \qquad k = 0,\ 1,\ 2,\ ...\ , \tag{23}$$

where

$$\tilde{b}_0 = \frac{1}{b_0}, \qquad \tilde{b}_k = -\frac{1}{b_0} \sum_{m=0}^{k-1} \tilde{b}_m b_{k-m}, \qquad k = 1,\ 2,\ ...$$

The corresponding *lower* commuting flows (in Riemann invariants) are

$$\partial_{t_{-k-1}} r^i = [\sum_{m=0}^{k} \tilde{b}_m (r^i)^{m-k-1}] r_x^i, \qquad k = 0,\ 1,\ 2,\ ...\ , \tag{24}$$

If $\lambda \to \infty$, $\tilde{\lambda} \to \infty$, all the *higher* conservation laws for the *higher* commuting flows are

$$\partial_{t_k} a_m = \partial_x [\sum_{s=0}^{k} \tilde{a}_s a_{k+m-s}], \qquad k = 0,\ 1,\ 2,\ ... \tag{25}$$

If $\lambda \to 0$, $\tilde{\lambda} \to \infty$, all the *lower* conservation laws for the *higher* commuting flows are

$$\partial_{t_n} b_k = \partial_x [\sum_{s=0}^{k} b_s \tilde{a}_{n+s-k}], \quad k \leqslant n, \qquad \partial_{t_n} b_k = \partial_x [\sum_{s=0}^{n} \tilde{a}_s b_{k+s-n}], \quad k \geqslant n. \tag{26}$$

If $\lambda \to 0$, $\tilde{\lambda} \to 0$, all the *lower* conservation laws for *lower* commuting flows are

$$\partial_{t_{-n-1}} b_m = \partial_x [\sum_{k=0}^{n} \tilde{b}_k b_{n+m+1-k}], \qquad n = 0,\ 1,\ 2,\ ... \tag{27}$$

If $\lambda \to \infty$, $\tilde{\lambda} \to 0$, all the *higher* conservation laws for the *lower* commuting flows are

$$\partial_{t_{-n-1}} a_{m+1} = \partial_x [\sum_{k=0}^{m} a_s \tilde{b}_{n+s-m}],\ m \leqslant n, \ \ \partial_{t_{-n-1}} a_{m+1} = \partial_x [\sum_{k=0}^{n} \tilde{b}_s a_{m+s-n}],\ m \geqslant n. \tag{28}$$

All these above formulas can be easily checked by a direct calculations.

For instance, the initial system (13) has the generating function of conservation laws

$$\partial_t \mu = \partial_x[(\lambda - a_1)\mu], \tag{29}$$

where an infinite set of the *positive* (polynomial) conservation laws is

$$\partial_{t_1} a_k = \partial_x[a_{k+1} - a_1 a_k], \qquad k = 1,\, 2,\, ... \tag{30}$$

and an infinite set of the *negative* conservation laws is

$$\partial_{t_1} b_0 = \partial_x(-a_1 b_0), \qquad \partial_{t_1} b_k = \partial_x[b_{k-1} - a_1 b_k], \qquad k = 1,\, 2,\, ... \tag{31}$$

The second commuting flow (see (13) and (22))

$$r^i_{t_2} = [(r^i)^2 - \varepsilon r^i \sum_{m=0}^{N} r^m + \frac{\varepsilon^2}{2}(\sum_{m=0}^{N} r^m)^2 - \frac{\varepsilon}{2}\sum_{m=0}^{N}(r^m)^2] r^i_x, \qquad i = 1,\, 2,\, ...\, ,\, N$$

has the generating function of conservation laws (see (21))

$$\mu_{t_2} = \partial_x[(\lambda^2 - a_1\lambda + a_1^2 - a_2)\mu],$$

where an infinite set of the *positive* (polynomial) conservation laws is (see (25))

$$\partial_{t_2} a_k = \partial_x[a_{k+2} - a_1 a_{k+1} + (a_1^2 - a_2)a_k]. \tag{32}$$

an infinite set of the *negative* conservation laws is (see (26))

$$\begin{aligned} \partial_{t_2} b_0 &= \partial_x[b_0(a_1^2 - a_2)], \qquad \partial_{t_2} b_1 = \partial_x[b_1(a_1^2 - a_2) - a_1 b_0], \\ \partial_{t_2} b_k &= \partial_x[b_k(a_1^2 - a_2) - a_1 b_{k-1} + b_{k-2}], \qquad k = 2,\, 3,\, ... \end{aligned}$$

The first *negative* ($\tilde{\lambda} \to 0$) commuting flow is (see (24))

$$r^i_{t_{-1}} = \frac{1}{b_0 r^i} r^i_x = \frac{\prod\limits_{m=1}^{N} (r^m)^\varepsilon}{r^i} r^i_x, \tag{33}$$

where the generating function of conservation laws is (see (23))

$$\mu_{t_{-1}} = \partial_x \frac{\mu}{\lambda b_0}, \tag{34}$$

where an infinite set of *negative* conservation laws is (see (27))

$$\partial_{t_{-1}} b_k = \partial_x \frac{b_{k+1}}{b_0}, \qquad k = 0,\, 1,\, 2,\, ... \tag{35}$$

and an infinite set of *positive* conservation laws is (see (28))

$$\partial_{t_{-1}} a_1 = \partial_x \frac{1}{b_0}, \qquad \partial_{t_{-1}} a_{k+1} = \partial_x \frac{a_k}{b_0}, \qquad k = 1,\, 2,\, ... \tag{36}$$

Remark 2 *Reciprocal transformation (see the first equation in (31))*

$$dy_{-1} = b_0 dx - a_1 b_0 dt_1, \qquad dz = dt_1,$$

connects system (13) (also, see (30) and (31)) and its commuting flow (33) (also, see (35) and (36)).

Reciprocal transformation (see (29))

$$dy = \mu[dx + (\lambda - a_1)dt_1], \qquad dz = dt_1$$

connects system (13) and generating function of its commuting flows (see (16)-(20))

$$\partial_t a_1 = -\lambda \partial_x (1/\mu).$$

3 New hydrodynamic chain

The hydrodynamic type system (13) can be rewritten as an infinite momentum chain

$$\partial_t c_k = \partial_x c_{k+1} - c_1 \partial_x c_k, \qquad k = 0, \ \pm 1, \ \pm 2, \ ... \tag{37}$$

where N first moments c_k ($k = 1, 2, ..., N$) are functionally independent

$$c_0 = \varepsilon \sum_{m=1}^{N} \ln r^m, \qquad c_k = \frac{\varepsilon}{k} \sum_{m=1}^{N} (r^m)^k, \qquad k = \pm 1, \ \pm 2, \ ... \tag{38}$$

Thus, all a_k, b_k and c_k can be expressed via each other (see below). This is invertible transformation.

However, now we can start our investigation namely from hydrodynamic chain written in form (37) or, for instance, (30) without reference on original hydrodynamic type system (13) (and explicit expressions (38)). If now we restrict our infinite momentum chain to $N-$component case, then just one particular solution obviously is the hydrodynamic type system (13) (also, see (38)). How to find all other possible reductions? The answer will be done in next section.

Remark 3 *For the first time, this hydrodynamic chain (30) has been derived (in another terms) by S.J. Alber (see [2]; also it has been independently obtained in another context by V.G. Mikhalev, see [26]), and recently by L.M. Alonso and A.B. Shabat (see [4]; they describe* ***mostly*** *differential reductions and* ***very particular*** *hydrodynamic reductions, in our article we describe* ***all possible*** *hydrodynamic reductions). Actually, the starting point of their investigations was the hydrodynamic type "$\varepsilon-$system", when $\varepsilon = -1/2$, related by generalized reciprocal transformation with averaged (by the Whitham method) integrable systems (determined by scalar second order spectral transform with energy-dependent potential, see also mentioned references) related with hyperelliptic surfaces.*

The generating function of conservation laws for the hydrodynamic chain (37) is exactly (29), where (sf. (14))

$$\mu=1+\sum_{k=1}^{\infty}a_k\lambda^{-k}=\exp[\sum_{k=1}^{\infty}c_k\lambda^{-k}],\qquad c_1\equiv a_1 \tag{39}$$

and $\lambda\to\infty$. The hydrodynamic chain (31) satisfies the same generating function (29), where (sf. (15))

$$\mu=\sum_{k=0}^{\infty}b_k\lambda^{k}=\exp[-\sum_{k=0}^{\infty}c_{-k}\lambda^{k}],\qquad c_0\equiv-\ln b_0 \tag{40}$$

and $\lambda\to 0$.

All the *positive* commuting flows in field variables c_k are

$$\partial_{t_n}c_k=\sum_{m=0}^{n}\tilde{a}_m c_{k+n-m,x},\qquad k=0,\ \pm 1,\ \pm 2,\ ...$$

where the generating function of conservation law densities for arbitrary *positive* commuting flow is (21); all *negative* flows are

$$\partial_{t_{-n-1}}c_k=\sum_{m=0}^{n}\tilde{b}_m c_{k+m-n,x},\qquad k=0,\ \pm 1,\ \pm 2,\ ...\ ,$$

where the generating function of conservation law densities for arbitrary *negative* commuting flow is (23). All these *negative* flows can be obtained from *positive* commuting flows (see above) by the reciprocal transformation (see **Remark** of previous section). For instance, the first *negative* flow is

$$\partial_{t_{-1}}c_k=e^{c_0}\partial_x c_{k-1},\qquad k=0,\ \pm 1,\ \pm 2,\ ... \tag{41}$$

Remark 4 *Obviously, values $\tilde{a}_k$ and $\tilde{b}_k$ can be expressed from analogues of (39) and (40) (see (20))*

$$\begin{aligned}\frac{1}{\mu}&=1+\sum_{k=1}^{\infty}\tilde{a}_k\lambda^{-k}=\exp[-\sum_{k=1}^{\infty}c_k\lambda^{-k}],\qquad \tilde{a}_1\equiv -c_1,\quad \lambda\to\infty.\\ \frac{1}{\mu}&=\sum_{k=0}^{\infty}\tilde{b}_k\lambda^{k}=\exp[\sum_{k=0}^{\infty}c_{-k}\lambda^{k}],\qquad \tilde{b}_0\equiv e^{c_0},\qquad \lambda\to 0.\end{aligned}$$

Thus, hydrodynamic chain (37) can be expressed via different moments (a_k, b_k), see, for instance Remark in section 6.

Relationship (39) *positive* moments c_k and *positive* conservation law densities a_k can be expressed explicitly by next four recursive formulas, where the first of them

$$da_{k+1}=\sum_{m=1}^{k}a_m dc_{k+1-m}+dc_{k+1},\qquad k=0,1,2,...$$

is consequence of three local symmetry operators acting on space of conservation law densities a_k: the shift operator

$$\hat{\delta}a_{k+1} = \frac{\partial a_{k+1}}{\partial c_1} = a_k, \qquad k = 0, 1, 2, ...,$$

the scaling operator

$$\hat{R}a_k = \sum_{m=1}^{\infty} mc_m \frac{\partial a_k}{\partial c_m} = ka_k, \qquad k = 0, 1, 2, ...$$

and projective operator

$$\hat{S}a_k = [c_1 + \sum_{m=1}^{\infty}(m+1)c_{m+1}\frac{\partial}{\partial c_m}]a_k = (k+1)a_{k+1}, \qquad k = 0, 1, 2, ...$$

4 Finite-component reductions

Theorem 5 *The hydrodynamic type system (8) with an arbitrary number of components is embedded into the hydrodynamic chain (37) if and only if*

$$V_i = f_i(r^i) - c_1, \qquad c_1 = \sum_{m=1}^{N} \psi_m(r^m), \tag{42}$$

where $f_i(r^i)$ and $\psi_k(r^k)$ are arbitrary functions.

Proof. Any reductions are compatible with given hydrodynamic chain (37) if every c_k can be expressed as a function of just N independent Riemann invariants r^i. Then one obtains

$$V_i \partial_i c_k = \partial_i c_{k+1} - c_1 \partial_i c_k, \qquad i = 1,\ 2,\ ...,\ N, \qquad k = 0,\ \pm 1,\ \pm 2,\ ...$$

It is easy to see, that

$$\partial_i c_{k+1} = (V_i + c_1)^k \partial_i c_1, \qquad i = 1,\ 2,\ ...,\ N, \qquad k = 0,\ \pm 1,\ \pm 2,\ ...$$

Thus, the second derivatives

$$\partial_j[(V_i + c_1)^k \partial_i c_1] = \partial_i[(V_j + c_1)^k \partial_j c_1], \quad i = 1,\ 2,\ ...,\ N, \qquad k = 0,\ \pm 1,\ \pm 2,\ ...$$

yield the general reduction (42). ■

Thus, all moments are

$$c_k = \sum_{m=1}^{N} \int^{r^m} [f_m(\lambda)]^{k-1} d\psi_m(\lambda), \qquad k = 0,\ \pm 1,\ \pm 2,\ ...$$

Remark 6 *The hydrodynamic type systems*

$$r^i_{t_{-1}} = W^i(\mathbf{r})r^i_x, \qquad i = 1,\ 2,\ \dots\ ,\ N \tag{43}$$

embedded into first negative flow (41) (see (37)) can be found in the same way (see (42)): from $W^i\partial_i c_k = e^{c_0}\partial_i c_{k-1}$ *one can obtain* $\partial_i c_k = (\frac{e^{c_0}}{W^i})^k \partial_i c_0$*. In comparison with* $\partial_i c_k = (V_i + c_1)^k \partial_i c_0$ *one can obtain*

$$W^i = \frac{1}{f_i(r^i)} \exp[\sum_{m=1}^{N} \int^{r^m} \frac{d\psi_m(\lambda)}{f_m(\lambda)}]. \tag{44}$$

Next task is how to write hydrodynamic reductions in *closed* form via special variables like conservation law densities (see, e.g. (30)). It means, that all higher moments a_{N+k} must be expressed via lower moments a_k ($k = 1,\ 2,\ \dots\ ,\ N$). In particular case (13), all higher moments a_{N+k} are polynoms, which can be found from relations (see (14))

$$0 = [(1 + \lambda a_1 + \lambda^2 a_2 + \dots + \lambda^N a_N + \lambda^{N+1} a_{N+1} + \dots)^{-1/\varepsilon}]^{(N+k)}, \qquad k = 1,\ 2,\ \dots$$

For example, the first higher moment a_{N+1} can be found from more compact recursive relation

$$\sum_{k=1}^{N+1} \frac{\Gamma(1 - 1/\varepsilon) z_k^{(N+1)}}{\Gamma(1 - k - 1/\varepsilon)\Gamma(k+1)} = 0,$$

where $z_k^{(N+1)}$ is coefficient of series

$$[a_1 + a_2\lambda + a_3\lambda^2 + \dots + a_{N+2-k}\lambda^{N+1-k}]^k = a_1^k + \lambda k a_1^{k-1} a_2 + \dots + \lambda^{N+1-k} z_k^{(N+1)} + \dots$$

All higher moments a_{N+k} are polynoms of lower moments a_k, e.g.:

$$a_3 = \tfrac{1+\varepsilon}{\varepsilon} a_1 a_2 - \tfrac{(1+\varepsilon)(1+2\varepsilon)}{6\varepsilon^2} a_1^3, \qquad N = 2,$$

$$a_4 = \tfrac{1+\varepsilon}{2\varepsilon}(2a_1a_3 + a_2^2) - \tfrac{(1+\varepsilon)(1+2\varepsilon)}{2\varepsilon^2} a_1^2 a_2 + \tfrac{(1+\varepsilon)(1+2\varepsilon)(1+3\varepsilon)}{24\varepsilon^3} a_1^4, \qquad N = 3,$$

$$a_5 = \tfrac{1+\varepsilon}{\varepsilon}(a_1a_4 + a_2a_3) - \tfrac{(1+\varepsilon)(1+2\varepsilon)}{2\varepsilon^2}(a_1^2 a_3 + a_1 a_2^2) +$$

$$\tfrac{(1+\varepsilon)(1+2\varepsilon)(1+3\varepsilon)}{6\varepsilon^3} a_1^3 a_2 - \tfrac{(1+\varepsilon)(1+2\varepsilon)(1+3\varepsilon)(1+4\varepsilon)}{120\varepsilon^4} a_1^5, \qquad N = 4,$$

and so on... First exceptional case, is chromatography phenomena ($\varepsilon = -1$), then

$$a_3 = -a_1a_2 + \tfrac{1}{6}a_1^3, \qquad N = 2,$$

$$a_4 = -\tfrac{1}{2}(2a_1a_3 + a_2^2) + \tfrac{1}{2}a_1^2a_2 - \tfrac{1}{12}a_1^4, \qquad N = 3,$$

$$a_5 = -(a_1a_4 + a_2a_3) + \tfrac{1}{2}(a_1^2a_3 + a_1a_2^2) - \tfrac{1}{3}a_1^3a_2 + \tfrac{1}{20}a_1^5, \qquad N = 4;$$

the second exceptional case is (dispersionless limits of Coupled KdV and Coupled Harry Dym) $\varepsilon = -1/2$, then first higher moment a_{k+1} is just quadratic expression via lower moments a_k. It means, that corresponding hydrodynamic type systems have at least one local Hamiltonian structure (see [34]); actually, the most number of local Hamiltonian structures is $(N+1)$, *iff* $\varepsilon = -1/2$, see [15])). The third exceptional case is $\varepsilon = -1/M$, where $M = 3,\ 4,\ \dots$ Then all higher moments will be quickly trancated (see above).

5 Commuting flows and reductions

The corresponding linear system for conservation law densities (see [36]) is

$$\partial_{ik}h = \frac{\psi'_k(r^k)}{f_i(r^i) - f_k(r^k)}\partial_i h - \frac{\psi'_i(r^i)}{f_i(r^i) - f_k(r^k)}\partial_k h, \qquad i \neq k. \tag{45}$$

The general solution of such system is determined by N functions of a single variable (also, see [36]). In a particular, but very important, case of "$\varepsilon-$systems" (see, for example, (13)) this linear system (45) is exactly $N-$component generalization of the Euler-Darboux-Poisson system (see [29])

$$\partial_{ik}h = \frac{\varepsilon}{r^i - r^k}[\partial_i h - \partial_k h], \qquad i \neq k. \tag{46}$$

At first, it is necessary to find a generating function of conservation law densities μ, which can be found in comparison with (29) and (42)

$$\mu(\mathbf{r}, \lambda) = \exp \sum_{k=1}^{N} \int^{r^k} \frac{d\psi_k(\tilde{\lambda})}{\lambda - f_k(\tilde{\lambda})}. \tag{47}$$

This formula (47) simplifies in case of "$\varepsilon-$systems" (see (14) and (15), that was well known in case $N = 2$, for example, see [7]). Velocities w^i of the commuting flows (i.e. Riemann invariants r^i are considered as the functions simultaneously of three independent variables x, t, τ)

$$r^i_\tau = w^i(\mathbf{r})r^i_x, \qquad i = 1, 2, ..., N \tag{48}$$

can be found as the solutions of another linear system (also see [36])

$$\partial_i w^k = -\frac{\psi'_i}{f_i(r^i) - f_k(r^k)}(w^i - w^k), \qquad i \neq k. \tag{49}$$

Theorem 7 *Any solutions of the linear system (49) are connected with the solutions of another linear system (sf. (45))*

$$\partial_{ik}\tilde{h} = -\frac{\psi'_k(r^k)}{f_i(r^i) - f_k(r^k)}\partial_i \tilde{h} + \frac{\psi'_i(r^i)}{f_i(r^i) - f_k(r^k)}\partial_k \tilde{h}, \qquad i \neq k$$

by the differential substitution of the first order

$$w^i = \frac{1}{\psi'_i}\partial_i \tilde{h}.$$

Thus, the generating function of solutions (for the hydrodynamic type systems (8) and (42), (43) and (44); cf. (13) and (33)) by Tsarev generalized hodograph method (also see [36]) is

$$\begin{gathered} x + [f_i(r^i) - \sum_{k=1}^{N}\psi_k(r^k)]t_1 + \tfrac{1}{f_i(r^i)}\exp[\sum_{m=1}^{N}\int^{r^m}\tfrac{d\psi_m(\lambda)}{f_m(\lambda)}]t_{-1} = \\ -\tfrac{1}{\lambda - f_i(r^i)}\exp\left[-\sum_{k=1}^{N}\int^{r^k}\tfrac{d\psi_k(\tilde{\lambda})}{\lambda - f_k(\tilde{\lambda})}\right], \end{gathered} \tag{50}$$

where the generating function of commuting flows (see (19)) in Riemann invariants is

$$r^i_{\tau(\tilde{\lambda})} = \frac{1}{(1 - f_i(r^i)/\tilde{\lambda})\tilde{\mu}} r^i_x.$$

Remark 8 *If one selects monoms*

$$\psi_i(r^i) = \varepsilon_i r^i,$$

where ε_k are arbitrary constants, then the generalized "$\varepsilon-$systems"

$$r^i_t = [r^i - \sum_{m=1}^{N} \varepsilon_m r^m] r^i_x. \tag{51}$$

has the generating function of conservation law densities (cf. (14) and (47))

$$\mu = \prod_{m=1}^{N} (1 - r_m/\lambda)^{-\varepsilon_m}.$$

In general case (N is arbitrary) the hydrodynamic type system (51) is the natural $N-$parametric reduction of the hydrodynamic chain (37). When $N = 2$ this system (51) is natural two-parametric generalization of gas dynamics (9). If we choose (cf. (38))

$$c_0 = \sum_{m=1}^{N} \varepsilon_m \ln r^m, \qquad c_k = \frac{1}{k} \sum_{m=1}^{N} \varepsilon_m (r^m)^k, \qquad k = \pm 1,\ \pm 2,\ ...$$

then the hydrodynamic type system (51) satisfies for hydrodynamic chain (37).

Remark 9 *In the particular case $f_i(r^i) = \varepsilon_i$ (ε_i are arbitrary constants and $\varepsilon_i \neq \varepsilon_k$ for $i \neq k$; this is $N-$component generalization of gas dynamics, when the adiabatic index $\gamma = 1$) hydrodynamic type system (42)*

$$r^i_t = [\varepsilon_i - \sum_{m=1}^{N} \psi_m(r^m)] r^i_x$$

is "trivial" (also $\psi_i(r^i) \neq \mathrm{const}$). In this case, the linear system (45) has constant coefficients

$$\frac{\partial^2 h}{\partial R^i \partial R^k} = \frac{1}{\varepsilon_i - \varepsilon_k} [\frac{\partial h}{\partial R^i} - \frac{\partial h}{\partial R^k}], \qquad i \neq k,$$

where $R^i = \psi_i(r^i)$.

6 General solution

Description of the general solution for the linear system (45) is a very complicated task. Construction of the general solution has been made only in case of "$\varepsilon-$systems" (see [29]), when N is arbitrary; cases $N = 2$ and $N = 3$ were completely investigated by G. Darboux, L.P. Eisenhart and T.H. Gronwall (see [7] and [10]). The basic idea of how to construct a general solution (parameterized by N functions of a single variable, see [36]) of any over-determined linear systems like (45) was presented in [36] by recursive application of symmetry operators compatible with such systems. However, here we establish an alternative approach in the spirit of G. Darboux (see [36]; also, see the section concerning elliptic coordinates in [7]). Elliptic coordinates μ_α ($\alpha = 1, 2, ..., N$) appear in the theory of integrable hydrodynamic type systems associated with hyperelliptic curves, i.e. with "$\varepsilon-$systems", where $\varepsilon = -1/2$. G. Darboux suggested to introduce special variables r^k ($k = 1, 2, ..., N$) for separation of coordinates in Laplace equation by the following rule (see (15), when $\varepsilon = -1/2$)

$$\mu_\alpha^2 = \frac{\prod\limits_{k=1}^{N}(\gamma_\alpha - r^k)}{\prod\limits_{\beta\neq\alpha}(\gamma_\alpha - \gamma_\beta)}, \qquad g^{ii} = \frac{\prod\limits_{\beta=1}^{N}(r^i - \gamma_\beta)}{\prod\limits_{k\neq i}(r^i - r^k)},$$

where γ_α are arbitrary constants (the denominator $\prod\limits_{\beta\neq\alpha}(\gamma_\alpha - \gamma_\beta)$ in the first expression is just a constant multiplier, which does not affect the property to be a conservation law density, also see (29)) and flat (not constant) metric $g^{ii}(\mathbf{r})$ determined by

$$ds^2 = \sum_{\alpha=1}^{N}(d\mu_\alpha)^2 = \sum_{k=1}^{N} g_{kk}(dr^k)^2.$$

Thus, elliptic coordinates coincide with the Riemann invariants for "$\varepsilon-$systems", where $\varepsilon = -1/2$. It is easy to generalize Darboux coordinates μ_α to arbitrary ε

$$(\mu_\alpha)^{-1/\varepsilon} = \frac{\prod\limits_{k=1}^{N}(\gamma_\alpha - r^k)}{\prod\limits_{\beta\neq\alpha}(\gamma_\alpha - \gamma_\beta)}, \qquad \alpha = 1,\, 2,\, ...\, ,\, N.$$

In this case, all "$\varepsilon-$systems", for instance, (13) and (33) can be written explicitly via μ_α in the conservative form (see (30) and (35) in the particular case $\varepsilon = -1/2$)

$$\begin{aligned} \partial_t \mu_\alpha &= \partial_x \left[\left(\varepsilon \sum_{\beta=1}^{N} (\mu_\beta)^{-1/\varepsilon} + \gamma_\alpha - \varepsilon \sum_{\beta=1}^{N} \gamma_\beta \right) \mu_\alpha \right], \\ \partial_{t_{-1}} \mu_\alpha &= \frac{\prod\limits_{\beta=1}^{N} (\gamma_\beta)^\varepsilon}{\gamma_\alpha} \partial_x \left[\left(1 - \sum_{\beta=1}^{N} \frac{(\mu_\beta)^{-1/\varepsilon}}{\gamma_\beta} \right)^\varepsilon \mu_\alpha \right], \qquad \alpha = 1,\, 2,\, ...\, ,\, N. \end{aligned} \tag{52}$$

Remark 10 *Hydrodynamic type systems (13) and (33) for another set of moments* $E_k = \sum\limits_{\beta=1}^{N}(\gamma_\beta)^k(\mu_\beta)^{-1/\varepsilon}$ *(see (52)) can be written as the following hydrodynamic chains*

$$\partial_{t_1}E_k = \partial_x E_{k+1} + \varepsilon[E_0 - \sum_{\beta=1}^{N}\gamma_\beta]E_{k,x} - E_k E_{0,x}, \qquad k = 0,\ \pm 1,\ \pm 2,\ \dots\ , \tag{53}$$

$$\partial_{t_{-1}}E_k = \prod_{\beta=1}^{N}(\gamma_\beta)^\varepsilon[(1-E_{-1})^\varepsilon \partial_x E_{k-1} + (1-E_{-1})^{\varepsilon-1}E_{k-1}E_{-1,x}], \tag{54}$$

where

$$a_1 = \varepsilon(\sum_{\beta=1}^{N}\gamma_\beta - E_0), \qquad b_0 = \prod_{\beta=1}^{N}(\gamma_\beta)^{-\varepsilon}(1-E_{-1})^{-\varepsilon}.$$

Remark 11 *The hydrodynamic chain (53) is the same as the hydrodynamic chain (30), because these two chains are connected by invertible transformation (see (29) and (39))*

$$\mu^{-1/\varepsilon} = 1 + \sum_{k=0}^{\infty}E_k\lambda^{-(k+1)}, \tag{55}$$

where $\lambda \to \infty$*. The hydrodynamic chain (54) is the same as the hydrodynamic chain (35), because these two chains are connected by invertible transformation (see (34) and (40))*

$$\mu^{-1/\varepsilon} = 1 - \sum_{k=1}^{\infty}E_{-k}\lambda^{k-1},$$

where $\lambda \to 0$*.*

Thus, our approach is the following: we mark N arbitrary points $\lambda = \gamma_\alpha$ (N distinct punctures) on the Riemann surface $F(\lambda,\ \mu) = 0$ (see (47) and cf. (12)); then we obtain special set of coordinates

$$\mu_\alpha = \exp\sum_{k=1}^{N}\int_{\gamma_\alpha}^{r^k}\frac{d\psi_k(\tilde{\lambda})}{\gamma_\alpha - f_k(\tilde{\lambda})}, \tag{56}$$

which in fact is *fundamental basic of linearly independent solutions* for the corresponding linear system (45). It means that any solution of linear system (45) can be presented as a linear combination of the basic solutions (56) with some coefficients. Finally, we just mention, that any over-determined linear system like (45) must have a general solution which depends on N arbitrary parameters γ_α. In our case, we should take N infinite serieses of the conservation law densities $\mu_{\alpha,k}$ $(k = 1,\ 2,\ \dots)$ starting near already fixed punctures γ_α

$$\mu^{(\alpha)} = \mu_\alpha + (\lambda - \gamma_\alpha)\mu_{\alpha,1} + (\lambda-\gamma_\alpha)^2\mu_{\alpha,2} + \dots, \qquad \lambda \to \gamma_\alpha, \qquad \alpha = 1,\ 2,\ \dots\ ,\ N.$$

Thus, the general solution of linear system (45) is

$$h(\mathbf{r})=\sum_{\beta=1}^{N}\int\limits_{\gamma_\beta}^{r^\beta}\varphi_\beta(\lambda)\mu^{(\beta)}(\lambda)d\lambda,$$

where $\varphi_\beta(\lambda)$ are arbitrary functions, and the general solution of the hydrodynamic type system is given in an implicit form (see (8), (42), (43), (44) and (50)):

$$\begin{aligned} &x+[f_i(r^i)-\sum_{k=1}^{N}\psi_k(r^k)]t_1+\tfrac{1}{f_i(r^i)}\exp[\sum_{m=1}^{N}\int^{r^m}\tfrac{d\psi_m(\lambda)}{f_m(\lambda)}]t_{-1} \\ &=\tfrac{1}{\psi_i'(r^i)}\partial_i\left[\sum_{\beta=1}^{N}\int_{\gamma_\beta}^{r^\beta}\varphi_\beta(\lambda)\tilde{\mu}^{(\beta)}(\lambda)d\lambda\right], \end{aligned} \tag{57}$$

where

$$\begin{aligned} \tilde{\mu}_\alpha &= \exp\left[-\sum_{k=1}^{N}\int\limits_{\gamma_\alpha}^{r^k}\frac{d\psi_k(\tilde{\lambda})}{\gamma_\alpha-f_k(\tilde{\lambda})}\right], \qquad \tilde{\mu}(\mathbf{r},\lambda)=\exp\left[-\sum_{k=1}^{N}\int^{r^k}\frac{d\psi_k(\tilde{\lambda})}{\lambda-f_k(\tilde{\lambda})}\right], \\ \tilde{\mu}^{(\alpha)} &= \tilde{\mu}_\alpha+(\lambda-\gamma_\alpha)\tilde{\mu}_{\alpha,1}+(\lambda-\gamma_\alpha)^2\tilde{\mu}_{\alpha,2}+..., \qquad \lambda\to\gamma_\alpha, \qquad \alpha=1,\,2,\,...\,,\,N. \end{aligned}$$

The general solution of a linear system (45) can be presented in most possible explicit form in special case, when values ε are *integers* for "$\varepsilon-$systems". The case $N=2$ (namely Euler-Darboux-Poisson equation) was completely investigated (see, for instance, [35]). Its generalization on $N-$component case (46), or moreover on case of arbitrary *integers* ε_m (see (51))

$$\partial_{ik}h=\frac{1}{r^i-r^k}[\varepsilon_k\partial_i h-\varepsilon_i\partial_k h], \qquad i\neq k. \tag{58}$$

can be made in the same way as in [35]. For simplicity, here we shall restrict our consideration on case of (46) (see [29]).

The general solution of (46) (if $\varepsilon=\pm n, \quad n=1,\,2,\,...$) is

$$h_{(n)}=\sum_{k=1}^{N}\frac{d^n}{d(r^k)^n}\left[\frac{\varphi_k(r^k)}{\sum\limits_{m\neq k}(r^k-r^m)^n}\right], \qquad h_{(-n)}=\sum_{k=1}^{N}\int^{r^k}\varphi_k(\lambda)\prod_{m=1}^{N}(\lambda-r^m)^n d\lambda,$$

where $\varphi_k(r^k)$ are N arbitrary functions of a single variable (if we replace $\varphi_k(\lambda)\to\varphi_k^{(nN+1)}(\lambda)$ in second *negative* case, then all integrals can be expressed via finite number of derivatives only). Thus, indeed, these solutions are general (see [36]) for *positive* and *negative integers* ε (see (13) and (46)). The general solutions for (58) can be obtained by recursive application of Laplace transformations (see [11]) to above formulas (also, see [1]).

If ε is negative and $\varepsilon \neq -n, \quad n = 1,\ 2,\ ...\ ,$ then above-mentioned solution (right) easily to generalize

$$h_\varepsilon = \sum_{k=1}^{N} \int_{\gamma_k}^{r^k} \varphi_k(\lambda) \prod_{m=1}^{N} (\lambda - r^m)^{-\varepsilon} d\lambda,$$

where γ_k $(k = 1,\ 2,\ ...\ ,\ N)$ are arbitrary constants. If ε is positive and $\varepsilon \neq n, \quad n = 1,\ 2,\ ...\ ,$ then above-mentioned solution (left) easily to generalize just in case, when εN is integer, then

$$h_\varepsilon = \sum_{k=1}^{N} \oint_{C_k} \frac{\varphi_k(\lambda) d\lambda}{\prod\limits_{m=1}^{N} (\lambda - r^m)^\varepsilon},$$

where C_k $(k = 1,\ 2,\ ...\ ,\ N)$ are simple small contours surrounding the points $\lambda = r^k$ $(k = 1,\ 2,\ ...\ ,\ N)$. However, in general case (when ε is positive and εN is not integer) these contours C_k could not be closed on corresponding Riemann surface, because a sum of all phase shifts (for every point) will not be proportional to $2\pi M$, where M is some integer. For avoiding this problem one can introduce another set of contours-dumb-bell shaped figures $C_{k,\ k+1}$ surrounding every two neighbor points $\lambda = r^k$ and $\lambda = r^{k+1}$ $(k = 1,\ 2,\ ...\ ,\ N)$. So, integration must change sign twice from clockwise to anticlockwise, then every time phase shift will be 2π exactly. However, the number of contours must be equal to $N-1$, because in opposite case (if number is N) all contours became linearly dependent. Thus, in this general case a general solution of (46) parametrized by N arbitrary functions of a single variable is

$$h_\varepsilon = \sum_{k=1}^{N-1} \oint_{C_{k,\ k+1}} \frac{\varphi_k(\lambda) d\lambda}{\prod\limits_{m=1}^{N} (\lambda - r^m)^\varepsilon} + \int_{-\infty}^{0} \frac{\varphi_N(\lambda) d\lambda}{\prod\limits_{m=1}^{N} (\lambda - r^m)^\varepsilon},$$

where for simplicity we assume (without lost of generacy), that real parts of Riemann invariants r^k (branch points on a complex Riemann surface) are numerated as follows: $0 < \mathrm{Re}\, r^1 < \mathrm{Re}\, r^2 < ... < \mathrm{Re}\, r^N$.

7 (2+1)-integrable hydrodynamic type systems

Benney momentum chain (1) is equivalent to the hierarchy of $(2+1)$ hydrodynamic type systems embedded in dispersionless KP hierarchy as Khohlov-Zabolotskaya equation

$$(u_{t_2} - u u_x)_x = u_{t_1 t_1}, \tag{59}$$

which can be obtained from the coupled equations of the Benney momentum chain (1) and one equation of its first nontrivial commuting flow (see the next section)

$$\partial_{t_2} A_k = \partial_x A_{k+2} + A_0 A_{k,x} + (k+1) A_k A_{0,x} + k A_{k-1} A_{1,x}, \qquad k = 0,\ 1,\ 2,\ ... \tag{60}$$

by eliminating moments A_1 and A_2:

$$\partial_{t_1}A_0=\partial_x A_1,\qquad \partial_{t_1}A_1=\partial_x[A_2+\frac{1}{2}A_0^2],\qquad \partial_{t_2}A_0=\partial_x[A_2+A_0^2],$$

where $u=A_0$.

Let us start now with the hydrodynamic chains (30) and (32), eliminate field variable a_3 from couple equations from first hydrodynamic chain (30) and one equation from the second hydrodynamic chain (60)

$$\partial_{t_1}a_1=\partial_x[a_2-a_1^2],\qquad \partial_{t_1}a_2=\partial_x[a_3-a_1a_2],\qquad \partial_{t_2}a_1=\partial_x[a_3-2a_1a_2+a_1^3].$$

Then we came to a new integrable $(2+1)$ hydrodynamic type system

$$u_{t_1}=w_x,\qquad u_{t_2}=w_{t_1}+uw_x-wu_x, \tag{61}$$

where

$$u=a_1,\qquad w=a_2-a_1^2.$$

It is easy to check, that all possible hydrodynamic reductions of this system (61) (see, for instance, approach in [25])

$$r^i_{t_1}=\mu_i(\mathbf{r})r^i_x,\qquad r^i_{t_2}=\zeta_i(\mathbf{r})r^i_x,\qquad i=1,\,2,\,...\;N$$

are completely the same as those found already (see (42)), where

$$\zeta_i=f_i^2(r^i)-f_i(r^i)\sum_{k=0}^{N}\psi_m(r^m)+\frac{1}{2}\left(\sum_{k=0}^{N}\psi_m(r^m)\right)^2-\sum_{k=0}^{N}\int^{r^k}f_k(\lambda)d\psi_k(\lambda).$$

Moreover, one can obtain a whole hierarchy of such integrable $(2+1)$ hydrodynamic type systems like (61) by eliminating some other field variables a_k in combination with another commuting flows of hydrodynamic chain (30). For example, two another equations (see (37) and (41))

$$\partial_{t_1}e^{-c_0}=\partial_x[-c_1e^{-c_0}],\qquad \partial_{t_{-1}}c_1=\partial_x e^{c_0} \tag{62}$$

yield a new integrable $(2+1)$ hydrodynamic type system (its $(1+1)$ hydrodynamic reductions are exactly).

8 Another hydrodynamic chain

Now we start with the integrable hydrodynamic type system [27]

$$r^i_t=\left[\sum_{m=1}^{N}\varepsilon_m r^m-\varepsilon_i\sum_{m=1}^{N}r^m\right]r^i_x,\qquad i=1,\,2,\,...\;,N, \tag{63}$$

when ε_k are arbitrary constants. This system can be rewritten as the hydrodynamic chain

$$\partial_t c_k=c_1\partial_x c_k-c_0\partial_x c_{k+1},\qquad k=0,\;\pm 1,\;\pm 2,\,..., \tag{64}$$

where moments

$$c_k=\sum_{m=1}^{N}r^m(\varepsilon_m)^k.$$

Theorem 12 *Under the reciprocal transformation*

$$dz = \frac{1}{c_0}dx + \frac{c_1}{c_0}dt, \qquad dy = dt$$

this hydrodynamic chain linearizes

$$\partial_y c_k + \partial_z c_{k+1} = 0, \qquad k = 0, \pm 1, \pm 2, \ldots \tag{65}$$

It means that any reductions such as (63) of the hydrodynamic chain (64) linearizes under above reciprocal transformation. The solution of the hydrodynamic chain (65) is a set of the separate Riemann-Monge-Hopf equations

$$r^i_y + f_i(r^i)r^i_z = 0, \qquad i = 1, 2, \ldots, N,$$

where $f_i(r^i)$ are arbitrary functions. Thus, every integrable reduction of hydrodynamic chain (64) has the simple form

$$r^i_t = [c_0 f_i(r^i) - c_1]r^i_x,$$

where

$$c_k = \sum_{m=1}^{N} \int^{r^m} [f_m(\lambda)]^k d\psi_m(\lambda), \qquad k = 0, \pm 1, \pm 2, \ldots$$

and $\psi_m(r^m)$ are arbitrary functions (by scaling $\psi_m(r^m) \to R_m$ integrable hydrodynamic reductions are parameterized by N arbitrary functions of a single variable only).

9 Open Problems

The Benney momentum chain (the Zakharov reduction) is a dispersionless limit of the vector nonlinear Shrodinger equation (see (11), (12) and [38]). The inverse problem is: how to reconstruct a *dispersive* integrable analogue of given hydrodynamic type system. A *dispersive* analogue is known (Coupled KdV is a *dispersive* analogue of system (13); Couple Harry Dym is a *dispersive* analogue of system (33), see, i.g. [20] and [15]) just in case of "$\varepsilon-$systems" with $\varepsilon = -1/2$. The KP hierarchy is a *dispersive* analogue for the *whole* Benney momentum chain (1) (as KP equation is a *dispersive* analogue of the Khohlov-Zabolotzkaya system (59)), but similar *dispersive* $(2+1)$ analogues for the whole hydrodynamic chain (37) or for $(2+1)$ hydrodynamic type systems (61) or (62) still are unknown.

Local Hamiltonian structures for the hydrodynamic type system (13) were completely investigated in [32] and [15]. It was proved, that if $N = 2$, then the hydrodynamic type system (13) for any ε has three local Hamiltonian structures (also, see [28] and [14]); if $N = 3$ and $\varepsilon = -1/2$, then it has four local Hamiltonian structures; if $N = 3$ and $\varepsilon = 1$, then it has two-parametric family of local Hamiltonian structures (also see [33] and [16]); if $N > 3$, then $\varepsilon = -1/2$ and it has $(N + 1)-$ local Hamiltonian structures.

Hamiltonian structures of integrable hydrodynamic type systems are determined by a metric g_{ii} (see, [9]). The metric

$$g^{ii}=\zeta_i(r^i)\exp\left[-2\sum_{k\neq i}\int^{r^k}\frac{d\psi_k(\lambda)}{f_i(r^i)-f_k(\lambda)}\right]$$

with an arbitrary functions $\zeta_i(r^i)$ determine a nonlocal Hamiltonian formalism (see [12] and [36]) of hydrodynamic type systems (8), (42), (43), (44) and their commuting flows. Unfortunately, local and nonlocal Hamiltonian formalism has been done just for the hydrodynamic type systems (13) when $\varepsilon=\pm 1$ and $\varepsilon=-1/2$ (see for instance [16] and [12]). However, the problem of a description of local and nonlocal Hamiltonian structures in general case (42) still is open. Nevertheless, this problem can be solved by the Dirac restriction of a Hamiltonian structure (see for the beginning [12]) known for the whole hydrodynamic chain, as it was already done in [6] for another hydrodynamic chains.

Starting point of such investigation is a Lax-like representation. For instance, the Lax-like representation (see (2)) for the dispersionless KP hierarchy (i.e. the Benney momentum chain (1)) is well known (see [24])

$$\partial_{t_n}\lambda=\{Q_n,\,\lambda\}=\frac{\partial Q_n}{\partial\mu}\frac{\partial\lambda}{\partial x}-\frac{\partial Q_n}{\partial x}\frac{\partial\lambda}{\partial\mu},\qquad n=0,\,1,\,2,\,...\ , \tag{66}$$

where Q_n is the part, polynomial in μ, of λ^n. Also, the first local Hamiltonian structure for whole Benney momentum chain (1)

$$\partial_{t_n}A_k=\sum_{m\geqslant 0}[kA_{k+m-1}\partial_x\frac{\delta H_{n+1}}{\delta A_m}+(mA_{k+m-1}\frac{\delta H_{n+1}}{\delta A_m})_x], \tag{67}$$

where the Hamiltonian is $H_2=\frac{1}{2}\int[A_2+A_0^2]dx$, was constructed in [23] (the relationship between formulas (66) and (67) was found in [24] too; first notrivial commuting flow (see (60)) is determined by the next Hamiltonian $H_3=\frac{1}{3}\int[A_3+3A_0A_1]dx$; functional $H_0=\int A_0dx$ is a Casimir of this Hamiltonian structure, the functional $H_1=\int A_1dx$ is a momentum of this Hamiltonian structure).

Similar Lax-like representation for the hydrodynamic chain (37) was established in [4] (cf. (66))

$$\partial_{t_n}L=\langle Q_n,\,L\rangle=Q_n\frac{\partial L}{\partial x}-\frac{\partial Q_n}{\partial x}L,\qquad n=0,\,1,\,2,\,...\ , \tag{68}$$

where

$$Q_n=(\lambda^nL)_+,\qquad L=1+G_0/\lambda+G_1/\lambda^2+G_2/\lambda^3+...$$

The corresponding first hydrodynamic chain

$$\partial_{t_1}G_k=\partial_xG_{k+1}+G_0G_{k,x}-G_kG_{0,x},\qquad k=0,\,1,\,2,\,...$$

is exactly the hydrodynamic chain (53), where $\varepsilon=1$, and linear term $-\varepsilon\left(\sum\limits_{\beta=1}^{N}\gamma_\beta\right)E_{k,x}$ is removed by shift of independent variable ($x\to x-\varepsilon\left(\sum\limits_{\beta=1}^{N}\gamma_\beta\right)t$). Thus, a generating

function of these moments is (see (53), (55) and (68))

$$\mu^{-1} \equiv L = 1 + \sum_{k=0}^{\infty} G_k \lambda^{-(k+1)}.$$

The alternative Lax-like representations are

$$\partial_{t_k}\rho = [\sum_{m=0}^{k} \tilde{a}_m \lambda^{k-m} \partial_x, \rho], \qquad \partial_{t_{-k-1}}\rho = [\sum_{m=0}^{k} \tilde{b}_m \lambda^{m-k-1} \partial_x, \rho], \qquad k = 0, 1, 2, \ldots ,$$

where $\mu = \rho_x$ (see (21), (23)).

We suppose that the hydrodynamic chain (37) and its commuting flows have local Hamiltonian structure (sf. (67))

$$\partial_{t_n} c_k = \sum_{k=1}^{\infty} [\beta_{k,m}(\mathbf{c})\partial_x + \partial_x \beta_{m,k}(\mathbf{c})] \frac{\delta H_{n+1}}{\delta c_m},$$

where $\beta_{k,m}$ are some functions.

The Hamiltonian structures of integrable hydrodynamic type systems can be successfully investigated by application of methods from the differential (see, for example, [36], [29]–[33], [14], [15], [12] and [16]) algebraic geometry (see, for instance, [8] and [22]). An alternative way is following: assume that our given integrable hydrodynamic type system (N components) is a some reduction of some "bigger" integrable hydrodynamic type system ($N+M$ components); assume that Hamiltonian structure of such "bigger" integrable hydrodynamic type system is already known. Then the direct application of the Dirac restriction to this Hamiltonian structure (the first step in such procedure (see [12]) is the choice of some Riemann invariants $r^k = \text{const}$, $k = 1, 2, \ldots , M$) yields the transformed Hamiltonian structure of a "restricted" hydrodynamic type system. The Dirac restriction of Hamiltonian structures (in algebraic language) was developed in application to hydrodynamic chains and their reductions (see [6]). The first step in such procedure is the recalculation of the Lax-like representation (such (66) or (68)) to the Hamiltonian structure of a whole hydrodynamic chain (see for instance (67)).

10 Conclusion

In this article we present a recipe: how to construct a hydrodynamic chain starting from any given hydrodynamic type system with *polynomial* (or *rational*) velocities with respect to their field variables (for simplicity we have mentioned just two cases: namely Benney momentum chain, whose moments are connected directly with some *conservation law densities* (u_k, η_k) and the hydrodynamic chain (37), whose moments are connected directly with *Riemann invariants*). In fact, it means that **any integrable hydrodynamic type system (written in Riemann invariants) with such *polynomial* velocities is embedded in hydrodynamic chain** (37) **or its higher (or lower) commuting flows.** Any integrable hydrodynamic type system written in Riemann invariants with

rational velocities

$$r_t^i = g_0 \frac{(r^i)^M + g_1(r^i)^{M-1} + ... + g_M}{(r^i)^K + e_1(r^i)^{K-1} + ... + e_K} r_x^i, \qquad i = 1,\, 2,\, ...,\, N \tag{69}$$

has a generating function of conservation laws

$$\mu_t = \partial_x [g_0 \frac{\lambda^M + g_1\lambda^{M-1} + ... + g_M}{\lambda^K + e_1\lambda^{K-1} + ... + e_K} \mu]. \tag{70}$$

For simplicity we assume that coefficients e_k and g_k of such *rational* velocities are some *symmetric* (not necessary to be *polynomials*!) functions of Riemann invariants; K, M and N are arbitrary natural numbers. As example, we can take any integrable systems embedded into $2x2$ spectral transform like Korteweg de Vries equation, Bonnet equation (Sin-Gordon equation) and nonlinear Shrodinger equation. All their Whitham deformations (i.e. hydrodynamic type systems, see, for instance, [8], [22]) have such representation (69) as consequence (70) written in abelian differentials on hyperelliptic surfaces

$$\partial_t dp = \partial_x dq,$$

where

$$dp = \frac{\lambda^K + e_1\lambda^{K-1} + ... + e_K}{\sqrt{\prod\limits_{m=1}^{N} (r_m - \lambda)}} d\lambda, \qquad dq = g_0 \frac{\lambda^M + g_1\lambda^{M-1} + ... + g_M}{\sqrt{\prod\limits_{m=1}^{N} (r_m - \lambda)}} d\lambda,$$

and velocities of (69) are

$$g_0 \frac{(r^i)^M + g_1(r^i)^{M-1} + ... + g_M}{(r^i)^K + e_1(r^i)^{K-1} + ... + e_K} = \frac{dq}{dp} \,|_{\lambda = r^i} \,.$$

Substituting (sf. (39))

$$\mu = 1 + \sum_{k=1}^{\infty} a_k \lambda^{-k}$$

into generating function (70) one can obtain similar formulas and results as it was done in this paper. Next step is *replication* of integrable hydrodynamic type systems as different hydrodynamic reductions of hydrodynamic chains. The *main advantage* of such *replicated* systems is *preservation* of some properties of original hydrodynamic systems like *generating functions of conservation laws and commuting flows* (see (2), (4) and (6) for Benney momentum chain (1); see (14), (29) and (20) for the hydrodynamic chain (37)). Thus, a problem of integrability is much simpler – all that necessary to do: to construct a general solution (starting from already obtained generating function, see (7), (50) and (57)), parameterized by N functions of a single variable (see [36]) and to solve Cauchy problem, that in fact is done to this moment just for four cases (the Zakharov reduction of the Benney momentum chain, see [17]; linearly degenerate system, see [37], this particular class is "$\varepsilon-$systems" with $\varepsilon = 1$; hydrodynamic type systems of the Tample class,

this particular class is "$\varepsilon-$systems" with $\varepsilon=-1$; Whitham hydrodynamic type systems related with hyperelliptic surfaces such averaged $N-$phase solutions of Korteweg de Vries equation (KdV) or nonlinear Shrodinger equation (NLS), it was done in articles of *G. El, T. Grava, B.A. Dubrovin, F.R. Tian, J. Gibbons* and many others).

Moreover, any two commuting flows of (29), for example (21)

$$dz=\mu[dx+\sum_{m=0}^{k}\tilde{a}_m\lambda^{k-m}dt^k+\sum_{m=0}^{n}\tilde{a}_m\lambda^{n-m}dt^n],\qquad k,n=1,2,...$$

yield hydrodynamic type systems with *rational* velocities

$$r_{t_k}^i=\frac{\sum\limits_{m=0}^{k}\tilde{a}_m(r^i)^{k-m}}{\sum\limits_{m=0}^{n}\tilde{a}_m(r^i)^{n-m}}r_{t_n}^i,\qquad k\neq n,\qquad i=1,\,2,\,...,\,N.$$

A more complicated *rational* dependence can be obtained by application of a generalized reciprocal transformation (see, for instance, [13] and [16]), starting from (29) and its commuting flows.

Thus, this is *powerful tool for classification* of integrable hydrodynamic type systems and their *integrability*. Moreover, if any given hydrodynamic type system has a generating function of conservation laws (see for instance, (4) or (29)), it means that the corresponding hydrodynamic chain has the same generating function. For example, if a some hydrodynamic type system has the same generating function as the Benney momentum chain (4), it means that this hydrodynamic type system is a some reduction of the Benney momentum chain. Thus, this is a wonderful symptom in recognition of an immersion of any unknown hydrodynamic type systems into already known hydrodynamic chains. Thus, *if one can construct a generating function of conservation laws for some hydrodynamic type system, it means that, in fact, hydrodynamic chain is already constructed* (and may be recognized, because, obviously, amount of hydrodynamic chains is much smaller than amount of integrable hydrodynamic type systems).

However, the problem of a description of all possible reductions is very complicated. For instance, this problem for the Benney momentum chain is still open (see [19]). However, this problem for the Boyer-Finley momentum chain (continuum limit of the Darboux-Laplace chain, which also is known as two-dimensional Toda lattice, see [3], [25] and [22]) in fact is *not exist*, because both mentioned hydrodynamic chains are related by special exchange of independent variables, see [21]). Thus, we are lucky to solve this problem for the hydrodynamic chain (37).

Acknowledgement 13 *I am grateful to my friends and colleagues Eugeni Ferapontov for his suggestions in improvement of my calculations and Yuji Kodama for useful discussions.*

References

[1] *A.V. Aksenov,* Symmetries and relations between solutions of a class of Euler-Poisson-Darboux equations. (Russian) Dokl. Akad. Nauk. (Reports of RAS) **381**, No. 2 (2001) 176–179.

[2] *S.J. Alber,* Associated integrable systems, J.Math.Phys., **32**, No. 4 (1991) 916-922.

[3] *E. Alfinito, G. Soliani, L. Solombrino*, The Symmetry Structure of the Heavenly Equation, Lett. Math. Phys. **41** (1997) 379–389.

[4] *L.M. Alonso, A.B. Shabat*, Energy-dependent potentials revisited: A universal hierarchy of hydrodynamic type, Physics Letters A, **299**, No. 4 (2002) 359-365.

[5] *D.J. Benney*, Some properties of long non-linear waves, Stud. Appl. Math., **52** (1973) 45-50.

[6] *M. Blaszak, B.M. Szablikowski*, Classical $R-$matrix theory of dispersionless systems. (1+1)–dimension theory. *to appear in* J. Phys. A: Math. Gen.

[7] *G. Darboux,* Leçons sur les systèmes orthogonaux et les coordonnées curvilignes, Paris (1910).

[8] *B.A. Dubrovin,* Hamiltonian formalism of Whitham-type hierarchies and topological Landau-Ginsburg models, Comm. Math. Phys., **145** (1992) 195-207.

[9] *B.A. Dubrovin, S.P. Novikov,* Hamiltonian formalism of one-dimensional systems of hydrodynamic type and the Bogolyubov-Whitham averaging method, Soviet Math. Dokl., **27** (1983) 665–669. *B.A. Dubrovin, S.P. Novikov,* Hydrodynamics of weakly deformed soliton lattices. Differential geometry and Hamiltonian theory, Russian Math. Surveys, **44**:6 (1989) 35–124.

[10] *L.P. Eisenhart,* Transformations of surfaces, Princeton (1923), 2nd ed.- Chelsea (1962). *T.H. Gronwall,* Annals of Math., **20** (1918/19) 262-263.

[11] *E.V. Ferapontov,* Laplace transformations of hydrodynamic type systems in Riemann invariants: periodic sequences, J.Phys.A: Math. Gen. **30** (1997) 6861-6878.

[12] *E.V. Ferapontov,* Nonlocal Hamiltonian operators of hydrodynamic type: differential geometry and applications, Amer. Math. Soc. Transl. (2), **170** (1995) 33-58.

[13] *E.V. Ferapontov*, Integration of weakly nonlinear hydrodynamic systems in Riemann invariants, Phys. Lett. A **158** (1991) 112-118.

[14] *E.V. Ferapontov,* Hamiltonian systems of hydrodynamic type and their realization on hypersurfaces of pseudo-Euclidean space. J.Soviet.Math. **55** (1991) 1970-1995.

[15] *E.V. Ferapontov, M.V. Pavlov*, Quasiclassical limit of coupled KdV equations, Riemann invariants and multi-Hamiltonian structure, Physica D, **52** (1991) 211-219.

[16] *E.V. Ferapontov, M.V. Pavlov*, Reciprocal transformations of Hamiltonian operators of hydrodynamic type: nonlocal Hamiltonian formalism for linearly degenerate systems. *accepted in* J.Math.Phys.

[17] *V. V. Geogjaev*, On the continuous Benney equations, Physica D **87** (1995) 168-175.

[18] *J. Gibbons*, Collisionless Boltzmann equations and integrable moment equations, Physica D, **3** (1981) 503-511.

[19] *J. Gibbons, S.P. Tsarev*, Reductions of Benney's equations, Phys. Lett. A, **211,** (1996) 19-24. *J. Gibbons, S.P. Tsarev,* Conformal maps and reductions of the Benney equations, Phys. Lett. A, **258** (1999) 263-270.

[20] *M. Jaulent and Y. Miodek*, Nonlinear evolution equations associated with energy-dependent Schrodinger potentials, Lett. Math. Phys. **1** (1975/77) 243-250. *A.P. Fordy*, Isospectral flows: their Hamiltonian structures, Miura maps and master symmetries, in: P.J. Olver, D.L. Sattinger (Eds.), Solitons in Physics, Mathematics, and Nonlinear Optics, Springer, New York (1990) 97–121.

[21] *Y. Kodama*, Solution of the dispersionless Toda equation, Phys. Lett. A, **147** (1990) 477-480. *L. Yu*, Waterbag reductions of the dispersionless discrete KP hierarchy, J. Phys. A: Math. Gen., **33** (2000) 8127–8138.

[22] *I.M. Krichever,* The dispersionless Lax equations and topological minimal models, Comm. Math. Phys., **143,** No. 2 (1992) 415-429. *I.M. Krichever,* The τ-function of the universal Whitham hierarchy, matrix models and topological field theories, Comm. Pure Appl. Math. **47** (1994) 437-475.

[23] *B.A. Kupershmidt, Yu.I. Manin*, Long wave equations with a free surface. II. The Hamiltonian structure and the higher equations. Func. Anal. Appl. **12,** No. 1 (1978) 25–37.

[24] *D.R. Lebedev, Yu.I. Manin*, Conservation laws and Lax representation of Benney's long wave equations, Phys. Lett. A, **74,** No.3,4 (1979) 154-156.

[25] *E.V. Ferapontov, D.A. Korotkin, V.A. Shramchenko,* Boyer-Finley equation and systems of hydrodynamic type. Class. Quantum Grav. **19** (2002) L1-L6.

[26] *V.G. Mikhalev,* Hamiltonian formalism of Korteweg-de Vries-type hierarchies, Funct. Anal. Appl., **26**, No. 2 (1992) 140-142.

[27] *O.I. Mokhov,* Symplectic and Poisson geometry on loop spaces of smooth manifolds and integrable equations, Review in Math. and Math.Phys., **11**, part. 2 (2002) Harwood Academic Publishers.

[28] *Y. Nutku*, On a new class of completely integrable nonlinear wave equations. Multi-Hamiltonian structure, J. Math. Phys., **28** (1987) 2579–2585.

[29] *M.V. Pavlov,* PhD Thesis, Landau ITP, Moscow, 1992.

[30] *M.V. Pavlov,* Elliptic coordinates and multi-Hamiltonian structures of hydrodynamic type systems, Reports of RAS, **339**, No. 1 (1994) 21-23.

[31] *M.V. Pavlov,* Exact integrability of a system of Benney equations. Russian Phys. Dokl., **39**, No. 11 (1994) 745–747. *M.V. Pavlov,* Local Hamiltonian structures of Benney's system. Russian Phys. Dokl., **39**, No. 9 (1994) 607-608.

[32] *M. V. Pavlov,* Hamiltonian formalism of electrophoresis equations. Integrable equations of hydrodynamics. London Inst. preprint 87-17 (1987). Moscow.

[33] *M. V. Pavlov,* Hamiltonian formalism of weakly nonlinear systems in hydrodynamics. Teor. Math. Phys. **73**, No. 2 (1987) 316–320.

[34] *M. V. Pavlov, S.P. Tsarev,* Three-Hamiltonian structures of the Egorov hydrodynamic type systems. accepted in Funct. Anal. Appl.

[35] *B.L. Rozhdestvenski, N.N. Yanenko*, Systems of quasilinear equations and their applications to gas dynamics. Translated from the second Russian edition by J. R. Schulenberger. Translations of Mathematical Monographs, 55. American Mathematical Society, Providence, RI, 1983; Russian ed. Nauka, Moscow (1968).

[36] *S.P. Tsarev*, On Poisson brackets and one-dimensional Hamiltonian systems of hydrodynamic type, Soviet Math. Dokl., **31** (1985) 488–491. *S.P. Tsarev*, The geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph method, Math. USSR Izvestiya **37**, No. 2 (1991) 397–419.

[37] *S.P. Tsarev*, Semi-Hamiltonian formalism of diagonal systems of hydrodynamic type and integrability of equations of chromatography and electrophoresis, preprint LOMI, St.Petersburg (1989) No. 106, 30-35. *S.P. Tsarev*, (in Russian) Integration of chromatography equations for Lengmur isoterm of absorption. in "Tomographic methods in physical-technological measurement, VNIIFTRI, (1985) 120-125.

[38] *V.E. Zakharov*, Benney's equations and quasi-classical approximation in the inverse problem method, Funct. Anal. Appl., **14**, No. 2 (1980) 89-98. *V.E. Zakharov*, On the Benney's Equations, Physica 3D (1981) 193-200.